

THE STRONG CONVERGENCE OF SCHRODINGER PROPAGATORS

BY

ALAN D. SLOAN¹

ABSTRACT. Time dependent versions of the Trotter-Kato theorem are discussed using nonstandard analysis. Both standard and nonstandard results are obtained. In particular, it is shown that if a sequence of generators converges in the strong resolvent topology at each time to a limiting generator and if the sequence of generators and limiting generator uniformly satisfy Kisynski type hypotheses then the corresponding Schrodinger propagators converge strongly. The results are used to analyze time dependent, form bounded perturbations of the Laplacian.

I. Introduction. The time dependent Schrodinger equation

$$(dx/dt)(t) = -iA(t)x(t), \quad x(s) = x_s, \quad (1)$$

may be advantageously discussed in terms of unitary propagators. Equation (1) is set in a complex Hilbert space, \mathcal{H} , and the time variables, t, s , are to range in some closed and possibly infinite interval, J , of real numbers. For each such t , $A(t)$ is a selfadjoint operator on \mathcal{H} while $x(t)$ is an element of \mathcal{H} as is x_s . $A(\cdot)$ is called the *generator* of (1).

DEFINITION. A jointly strongly continuous map $(t, s) \rightarrow U(t, s)$ from $J \times J$ into the unitary operators on \mathcal{H} is a *unitary propagator* providing

$$(a) \ U(t, t) = I,$$

$$(b) \ U(t, s)U(s, r) = U(t, s),$$

hold for all r, s, t in J .

Given a generator A the relevant propagator is expected to have the property that $x(t) = U(t, s)x_s$ is the "solution" to equation (1). The precise sense in which $x(\cdot)$ is a solution and the exact collection of initial states x_s for which such a solution exists remain to be specified later. At times, to avoid ambiguity, we will denote the propagator related to equation (1) by U_A .

The most satisfactory results are known in case $A(t) = A$ for each t in J where A is some fixed selfadjoint operator. This is the time independent case. The related propagator may be explicitly given as $U_A(t, s) = e^{i(t-s)A}$. Then $x(t) = U(t, s)x_s$ is differentiable in norm and satisfies equation (1) for all x_s in $D(A)$, the domain of A .

Once a solution is known to exist, an analysis of stability properties of the equation may begin; this is perturbation theory. The basic question in this theory is: If two generators are close, are the corresponding solutions close? In the time

Received by the editors July 6, 1979 and, in revised form, April 15, 1980.

AMS (MOS) subject classifications (1970). Primary 35B25, 35J10, 47D99, 81A10.

Key words and phrases. Evolution equations, propagators, perturbations, Schrodinger equation.

¹Research supported in part by NSF under Grant MCS76-07543.

independent case, this question is affirmatively answered by the

TROTTER-KATO THEOREM 1 [5, p. 502]. *If A_n is a sequence of selfadjoint operators whose resolvents converge strongly to the resolvent of a selfadjoint operator A , then $U_{A_n}(t, s)$ converges strongly to $U_A(t, s)$ for each t, s in J .*

The situation is more complicated in the time dependent case. For example, for t in \mathbf{R}^1 , let $f_n(t) = 1 + 2^n X_n(t)$, where X_n is the characteristic function of the interval $[n^{-1} - 2^{-(n+1)}, n^{-1} + 2^{-(n+1)}]$ for $n = 1, 2, \dots$. Let $A_n(t) = f_n(t)I$ and $A = I$. Then $A_n(t)$ converges in the strong resolvent topology to A but $U_{A_n}(t, 0)$ does not converge strongly to $U_A(t, 0)$ for any $t > 0$, even though the A_n 's and A are bounded.

In §II we present a Trotter-Kato type theorem for the case of bounded generators. In §III a similar result is presented for a certain class of unbounded generators. For unbounded and time dependent generators A , there is a general technique for constructing U_A . First one approximates A by bounded generators A_n , next one constructs U_{A_n} explicitly, and finally one proves that the U_{A_n} converge to a propagator which is then defined to be U_A .

Yoshida [15] developed an approximation procedure which has proved to be very durable. He defined $A_n(t)$ to be $A(t)[1 + A(t)/n]^{-1}$. This method has recently been applied in [16] to study the "hyperbolic case" of evolution equations. In addition to its continued applicability, another indication of the strength of the Yoshida approximation may be described in the framework of nonstandard analysis. In the next paragraph we informally discuss the ideas involved.

Given some set S one can form *S , a new set containing not only a copy of S but also ideal elements, whenever S is infinite. For example if \mathbf{N} is the set of positive integers then $^*\mathbf{N}$ contains infinite integers while for \mathbf{R} the reals, $^*\mathbf{R}$ contains both infinite and infinitesimal numbers. The map $V \rightarrow ^*V$ of subsets of S to subsets of *S is a Boolean algebra isomorphism into. Consequently, all the concepts of standard analysis on S may be transferred to *S . In particular there is a natural order on $^*\mathbf{R}$ and an induced definition of convergence: for a_n, a in $^*\mathbf{R}$, $n \in ^*\mathbf{N}$, $a_n \rightarrow a$ means for every $\varepsilon > 0$ in $^*\mathbf{R}$ there is an N in \mathbf{N} such that $n > N$ implies $|a_n - a| < \varepsilon$. Given such a sequence a_n one may also ask if $a_n \rightarrow a$ in the standard sense that for every $\varepsilon > 0$ in $\mathbf{R} \exists N$ in \mathbf{N} such that $n > N$ implies $|a_n - a| < \varepsilon$. These two types of convergence are not equivalent. For example, if m is an infinite integer then $a_n = m/n \rightarrow a = 0$ in the transferred sense, but not the standard sense.

In §III we consider the class K of generators introduced by Kisynski [8], and studied also by Simon [11]. The aforementioned authors showed that whenever A is in K , then U_{A_n} converges strongly to U_A where A_n is the Yoshida approximation to A . Transferring this result we obtain the fact that whenever B is in *K then U_{B_n} converges strongly to U_B . A priori, we may not assert that U_{B_n} converges to U_B in the standard sense. Nevertheless we are able to prove that this additional convergence takes place, giving an additional indication of the strength of the Yoshida approximation. Moreover, the general techniques of nonstandard analysis may be

used to show that in order to obtain a standard Trotter-Kato type theorem in K (i.e., whenever C_n, C are in K and $C_n(t) \rightarrow C(t)$ in the strong resolvent topology for each t , then $U_{C_n}(t, s)$ converges strongly to $U_C(t, s)$) the standard convergence of $U_{B_n}(t, s)$ to $U_B(t, s)$ for B in $*K$ is sufficient. Thus we are able to present a new standard result obtained first using nonstandard techniques. Our methods also revealed a standard proof which we present. The nonstandard results do have interesting consequences in §IV.

In §IV we are particularly interested in generators of the form $A(t) = -\Delta + V(t)$, selfadjoint on $L^2(\mathbf{R}^n)$ for each t . The corresponding propagator describes the evolution of the quantum system with Hamiltonian $A(t)$ at time t . If $V(t)$ is a quadratic form then the usual formulas of quantum physics do not have an obvious meaning; see for example the Dyson expansion, Theorem 3, which is typically the starting point for time dependent perturbation theory. Nonetheless, if one regularizes the $V(t)$ it is possible to obtain a nonstandard bounded $W(t)$ so that U_A and U_B are infinitely close, where $B(t) = -\Delta + W(t)$. Thus, the usual perturbation formulas, which are valid for U_B , may be used for U_A with only infinitesimal errors. Consequently, we obtain a nonstandard Dyson expansion for propagators of quantum mechanical systems of particles in potentials given as distributions. These and other applications are discussed further in §IV.

Finally, we remark that the reader interested in perturbation theory but not nonstandard analysis may read the remark following Theorem 7, the introduction to §III through the discussion of Theorem 8, Corollary 10, Theorems 11, 12, Corollary 13 and Examples 14 and 15 as these standard results are self-contained.

II. Stability of solutions for bounded generators. Let $*X$ be an adequate ultrapower of a structure X containing the real numbers, \mathbf{R}^1 .

For A selfadjoint and B an internal selfadjoint operator we write $A \approx B$ if B is in the monad of A relative to the strong resolvent topology. In this case we say that B is near standard and that A is the standard part of B . If P is a bounded linear operator and Q an internal finitely bounded linear operator write $P \approx Q$ in case $Ph \approx Qh$ for all h in \mathcal{H} . In case P and Q are additionally selfadjoint the two definitions of \approx coincide. Here we have written $h \approx k$ to mean $\|h - k\| \approx 0$ for h, k in $*\mathcal{H}$. This coincides with h being in the monad of k , in case $k \in \mathcal{H}$, relative to the norm topology on \mathcal{H} . The weak topology relation will be denoted by $h \approx_\omega k$. This means $(h, v) \approx (k, v)$ for all v in \mathcal{H} . On \mathbf{R} , \approx is with respect to the ordinary Euclidean topology.

In [13] it was shown that a standard technique of proving the Trotter-Kato theorem (see [5]) could be used to prove a nonstandard version:

THEOREM 2. *If A is selfadjoint and B is internal selfadjoint then $A \approx B$ implies $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in \mathbf{R}^1 .*

See [13] for a discussion of the strong resolvent topology from a nonstandard point of view. For an introduction to nonstandard analysis the reader is referred to [9] and [14].

DEFINITION. A *D-generator* is a strongly continuous map of J into the bounded selfadjoint operators.

An existence theorem for solutions of the Schrodinger equation defined by a *D-generator* is proved by explicit construction via the Dyson expansion.

THEOREM 3 [10]. Let A be a *D-generator*. Then defining $U(t, s)$ by

$$I + \sum_{n=1}^{\infty} (-i)^n \mathfrak{T}_n(A, t, s) \quad \text{for } t \geq s \text{ in } J,$$

where

$$\mathfrak{T}_n(A, t, s) = \int_s^t \int_s^{t_1} \int_s^{t_{n-1}} A(t_1) \cdots A(t_n) dt_n \cdots dt_1,$$

and by $U^*(s, t)$ for $s \geq t$ in J , gives a unitary propagator. Here the integrals are strong while the sum converges in operator norm. $x(t) = U(t, s)x_s$ gives a norm differentiable function which is a solution to equation (1) for all x_s in \mathcal{H} .

The main ingredient in the proof of Theorem 3 is an application of the uniform boundedness principle to prove

LEMMA 4 [10]. Let A be a *D-generator* and K a compact interval in J . Then $\sup_{t \in K} \|A(t)\| < \infty$.

LEMMA 5. Let A be a bounded linear operator and B an internal finitely bounded linear operator. If $g, h \in {}^*\mathcal{H}$, $g \approx h$ and $Ag \approx Bg$ then $Ah \approx Bh$.

PROOF.

$$\begin{aligned} \|Ah - Bh\| &\leq \|A(h - g)\| + \|(A - B)g\| + \|B(g - h)\| \\ &\leq \|A\| \|h - g\| + \|(A - B)g\| + \|B\| \|g - h\| \approx 0. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 6. Let A be a *D-generator*. Then for all finite t, s in *J , $t \approx s$ implies $A(t) \approx A(s)$. In particular $A(t)$ is near standard for each finite t in *J .

PROOF. If t, s in *J are finite there is an N in \mathbb{N} , the nonnegative integers, such that $|t| \leq N$ and $|s| \leq N$. By Lemma 4 $\sup_{t \in \mathbb{R}; |t| \leq N} \|A(t)\| = c$ is finite. By transfer, $\|A(t)\| \leq c$ for all $t \in {}^*\mathbb{R}$, $|t| \leq N$. If $s \approx t$, then $s \approx r \approx t$, $r \in \mathbb{R}$ and $|r| \leq N$. By strong continuity, $A(s)h \approx A(r)h \approx A(t)h$. Since $A(s), A(r), A(t)$ are all finitely bounded we obtain $A(s) \approx A(r) \approx A(t)$. Q.E.D.

THEOREM 7. Let A be a *D-generator* and B an internal *D-generator*. Suppose additionally that $\|B(t)\|$ is finite for all finite t in *J .

Then $A(t) \approx B(t)$ for all finite t in *J implies $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in *J .

PROOF. Fix t, s finite in *J . Let K be a compact interval in J so that t, s are in *K . It suffices to consider the case $t > s$.

B is internal so $\{N \in {}^*\mathbb{N} : \|B(r)\| \leq N \ \forall r \in {}^*K\}$ is internal and contains ${}^*\mathbb{N} - \mathbb{N}$. As ${}^*\mathbb{N} - \mathbb{N}$ is external we conclude that there is a c in \mathbb{N} such that $\|B(r)\| \leq c$ for all $r \in {}^*K$.

Write $\mathfrak{T}_n(\circ)$ for $\mathfrak{T}_n(\circ, t, s)$ and U_\circ for $U_\circ(t, s)$.

Choose α in ${}^*\mathbf{N} - \mathbf{N}$. Observe that

$$\left\| U_B - \left(I + \sum_{n=1}^{\alpha} (-i)^n \mathfrak{T}_n(B) \right) \right\| \leq \sum_{n=\alpha+1}^{\infty} \mathfrak{T}_n(\|B\|) \leq \sum_{n=\alpha+1}^{\infty} \frac{c^n (t-s)^n}{n!} \approx 0.$$

Similarly, $\|U_A - (I + \sum_{n=1}^{\alpha} (-i)^n \mathfrak{T}_n(A))\| \approx 0$. Consequently, we obtain

$$\begin{aligned} \|[U_B - U_A]h\| &\lesssim \sum_{n=1}^{\alpha} \int_s^t \int_s^{t_1} \cdots \\ &\quad \int_s^{t_{n-1}} \|[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)]h\| dt_n \cdots dt_1, \end{aligned} \quad (2)$$

for all h in ${}^*\mathcal{H}$ and α in ${}^*\mathbf{N} - \mathbf{N}$.

$$\begin{aligned} &B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n) \\ &= \sum_{i=1}^n B(t_1) \cdots B(t_{i-1}) [B(t_i) - A(t_i)] A(t_{i+1}) \cdots A(t_n). \end{aligned} \quad (3)$$

If the t_i 's are all finite and if h is near standard then by Lemmas 5 and 6 $A(t_{i+1}) \cdots A(t_n)h$ is near standard. By Lemma 5 $[B(t_i) - A(t_i)]A(t_{i+1}) \cdots A(t_n)h$ is some vector $h_{i,n}$ of infinitesimal norm, providing n is finite. Let $\delta_n = \max\{\|h_{i,n}\| : i = 1, 2, \dots, n\} \approx 0$, for n finite. From (3) we conclude

$$\|[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)]h\| \leq \sum_{i=1}^n c^{i-1} \|h_{i,n}\| \leq \delta_n (1 - c^n) / (1 - c).$$

Then

$$\|[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)]h\| \approx 0 \quad (4)$$

for all $n \in \mathbf{N}$ and t_i finite.

Fix $h \in \mathcal{H}$. Let $I_0 = \{m \in {}^*\mathbf{N} : (n \leq m, |t_1| \leq m, \dots, |t_n| \leq m) \rightarrow (\|m[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)]h\| \leq 1)\}$. I_0 is internal and contains, by (4) above, the external set \mathbf{N} . Thus $\gamma \in I_0$ for some γ in ${}^*\mathbf{N} - \mathbf{N}$. By (2) there now follows

$$\|(U_A - U_B)h\| \lesssim \sum_{i=1}^{\gamma} \frac{1}{\gamma} \frac{(t-s)^n}{n!} \leq \frac{e^{(t-s)}}{\gamma} \approx 0.$$

Thus, $U_A h \approx U_B h$ so that $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in *J . Q.E.D.

REMARK. If B_n is a sequence of standard generators with $\sup_n \sup_{t \in K} \|B_n(t)\| < \infty$ for all compact K , then the Dyson expansions for B_n converge to $U_{B_n}(t, s)$ uniformly in n for each t, s . Consequently, if $B_n(r)$ converges strongly to $A(r)$ then $\mathfrak{T}_m(B_n, t, s)$ converges to $\mathfrak{T}_m(A, t, s)$ for each m as $n \rightarrow \infty$ by dominated convergence and one may conclude $U_{B_n}(t, s)$ converges to $U_A(t, s)$ strongly. The previous proof follows this argument.

III. Stability of solutions for unbounded generators.

DEFINITION. A K -generator is a map A of J into the selfadjoint operator on \mathcal{H} for which there exist a selfadjoint operator $H \geq 1$ and a positive real number c such that

$$c^{-1}(H + 1) \leq A(t) \leq c(H + 1), \quad (5)$$

$$G(t) = \frac{d(A(t)^{-1})}{dt} \text{ exists in norm,} \quad (6)$$

$$\|A(t)^{1/2}G(t)A(t)^{1/2}\| \leq c, \quad (7)$$

hold for all t in J . H and c are called the *related constants*.

In [11] Simon presents an existence theorem for equations with K -generators. We have included the relevant part of this result as Theorem 8 below. Simon points out that, in fact, this theorem is a special case of a result due to Kisynski [8]. In this theorem there is no hypothesis concerning the independence of $D(A(t))$ from t . Thus it is particularly useful in treating problems of singular perturbations.

THEOREM 8. *Let A be a K -generator. There is a unitary propagator U such that defining $a(t) = U(t, s)x_s$ in $D(H^{1/2})$ gives a weak solution to Schrodinger's equation (1) in the sense that $(d/dt)(f, a(t)) = -i(A^{1/2}(t)f, A^{1/2}(t)a(t))$ for all f in $D(H^{1/2})$.*

SOME REMARKS ON THEOREM 8. Although the theorem only gives a weak solution it follows by a related uniqueness result that if a strong solution exists, it is the one given in Theorem 8.

Both Simon and Kisynski use a modification of techniques invented by Yoshida. One defines

$$A_n(t) = A(t)(1 + n^{-1}A(t))^{-1} \quad (8)$$

for $n = 1, 2, \dots$. Observing that $t \rightarrow A_n(t)$ is a D -generator and that

$$\|A_n(t)\| \leq n \quad (9)$$

for all t in J , one obtains a strong solution $a_n(t)$ to the Schrodinger equation (1) with generator A_n , using Theorem 3. One then proves

$$\lim_{n \rightarrow \infty} (f, a_n(t)) = (f, a(t)) \quad (10)$$

for all f in \mathcal{H} and t in J . Defining

$$\|g\|_{-1} = \|(H + 1)^{-1/2}g\|, \quad \|g\|_{+1} = \|(H + 1)^{1/2}g\|$$

it can be shown that for any compact interval T containing s , there is a constant K_T such that

$$\left| \frac{da_n(t)}{dt} \right|_{-1} \leq K_T \quad (11)$$

and

$$\|a_n(t)\|_{+1} \leq K_T \quad (12)$$

for all t in T . Here if τ is the largest t in T , then

$$K_T \leq (c + 1)^2 e^{3c(\tau-s)/2} \|x_s\|_{+1}. \quad (13)$$

The proofs of all these numbered facts may be found in [11].

THEOREM 9. *Let A be a K -generator with constants H, c . Let B be an internal K -generator with constants H, c .*

*If $A(t) \approx B(t)$ for all t in J then $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in $*J$.*

COROLLARY 10. *Let A_n, A be K -generators on J with constants (H, c) for $n = 1, 2, \dots$. Suppose $A_n(t)$ converges in the strong resolvent topology to $A(t)$ for each t in J . Then $U_{A_n}(t, s)$ converges strongly to $U_A(t, s)$ for every t and s in J .*

To obtain Corollary 10 from Theorem 9 one may proceed as follows. Choose $B(t) = A_n(t)$ for some positive infinite integer n . Equation (7) implies that $\|dB^{-1}(t)/dt\| \leq c^2$ for all t in $*J$. The norm mean value theorem then yields $\|B^{-1}(t) - B^{-1}(s)\| \approx 0$ whenever $t \approx s$, similarly for A^{-1} . Since we are assuming $A(s) \approx B(s)$ for each s in J we arrive at the conclusion $B^{-1}(t) \approx B^{-1}(s) \approx A^{-1}(s) \approx A^{-1}(t)$ whenever s is in J and $t \approx s$, using the fact that J is closed. Since, however, Corollary 10 is an interesting standard result, such a proof is presented. Both the proof of Theorem 9 and the corollary are based on the following standard computation.

TECHNICAL LEMMA. *Let T be any compact interval of \mathbf{R} and let K_T be as in equations (10), (11), (13). Then for all A in $K(H, c)$, $t > s$ in T , and x in $D(H^{1/2})$*

$$\|A^{-1/2}(t)(U_{A_n}(s, t) - U_{A_m}(s, t))x\| \leq 8K_T^2(e^{c(t-s)} - 1)\left(\frac{1}{m} + \frac{1}{n}\right)/c.$$

Here A_n is defined as in (8).

PROOF. Fix s in T and for $t > s$ in T define $a_n(t) = U_{A_n}(s, t)x$. Fix positive integers m and n . Define

$$p(t) = a_n(t) - a_m(t), \quad w(t) = (p(t), A^{-1}(t)p(t)),$$

and

$$(dA^{-1}/dt)(t) = G(t).$$

In general we denote df/dt by \dot{f} so that $\dot{p} = -iA_n a_n + iA_m a_m$ and $\dot{w} = (\dot{p}, A^{-1}p) + (p, Gp) + (p, A^{-1}\dot{p})$.

Since $A^{-1} = A_n^{-1} - 1/n$ we may write

$$\begin{aligned} \dot{w} &= (-iA_n a_n + iA_m a_m, A^{-1}p) + (p, Gp) + (p, A^{-1}(-iA_n a_n + iA_m a_m)) \\ &= i(A_n a_n, (A_n^{-1} - 1/n)p) - i(A_m a_m, (A_m^{-1} - 1/m)p) + (p, Gp) \\ &\quad + i(p, (A_m^{-1} - 1/m)A_m a_m) - i(p, (A_n^{-1} - 1/n)A_n p) \\ &= i(a_n, p) - (i/n)(A_n a_n, p) - i(a_m, p) + (i/m)(A_m a_m, p) + (p, Gp) \\ &\quad - i(p, a_n) + (i/n)(p, A_n a_n) + i(p, a_m) - (i/m)(p, A_m a_m) \\ &= i(a_n - a_m, p) - i(p, a_n - a_m) + (p, Gp) + (1/m)(p, -iA_m a_m) \\ &\quad - (1/n)(-iA_n a_n, p) + (1/m)(-iA_m a_m, p) - (1/n)(p, -iA_n a_n) \\ &= i(p, p) - i(p, p) + (p, Gp) - (1/n)(\dot{a}_n, p) \\ &\quad + (1/m)(\dot{a}_m, p) - (1/n)(p, \dot{a}_n) + (1/m)(p, \dot{a}_m). \end{aligned}$$

Thus

$$\dot{w} = (p, Gp) + (2/m) \operatorname{Re}(\dot{a}_m, p) - (2/n) \operatorname{Re}(\dot{a}_n, p). \quad (14)$$

Next observe that since

$$\begin{aligned} |(p, Gp)| &= |(A^{-1/2}p, A^{1/2}GA^{1/2}A^{-1/2}p)| \leq \|A^{1/2}GA^{1/2}\| \|A^{-1/2}p\|^2 \\ &= \|A^{1/2}GA^{1/2}\| (p, A^{-1}p) \end{aligned}$$

we obtain

$$|(p, Gp)| \leq cw. \quad (15)$$

Also, for r in \mathbf{N} , $|(\dot{a}_r, p)| = |((H+1)^{-1/2}\dot{a}_r, (H+1)^{1/2}p)| \leq \|\dot{a}_r\|_{-1} \|p\|_{+1} \leq \|\dot{a}_r\|_{-1} (\|a_n\|_{+1} + \|a_m\|_{+1})$, so that from (11), (12) and (13) there follows

$$|(\dot{a}_r(t), p(t))| \leq 2K_T^2. \quad (16)$$

Substituting (15) and (16) into (14) we find

$$|(dw/dt)(t)| \leq cw(t) + K(1/n + 1/m) \quad (17)$$

for all $t > s$ in T . Here K is the finite constant $8K_T^2$.

To finish the proof we observe that

$$\begin{aligned} w(s) &= (p(s), A^{-1}(s)p(s)) = (a_n(s) - a_m(s), A^{-1}(s)p(s)) \\ &= (x - x, A^{-1}(s)p(s)) = 0. \end{aligned}$$

Since $w \geq 0$, (17) implies

$$0 \leq w(t) \leq (K/c)(1/n + 1/m)e^{c(t-s)} - (K/c)(1/n + 1/m). \quad \text{Q.E.D.}$$

PROOF OF THEOREM 9. As the argument following Corollary 10 shows, $A(t) \approx B(t)$ for all finite t in $*J$. Fix a finite s in $*J$ and x in $D(H^{1/2})$. Define a_n as in the proof of the Technical Lemma and a as in Theorem 8. Similarly, define $B_n(t) = B(t)(1 + n^{-1}B(t))^{-1}$ so that B is an internal D -generator with

$$\|B_n(t)\| \leq n. \quad (18)$$

Let b_n be the solution to the Schrodinger equation (1) with generator B_n and initial state x . Since $A_n(t)^{-1} = A(t)^{-1} + 1/n \approx B(t)^{-1} + 1/n = B_n(t)^{-1}$ it follows from bounded stability, Theorem 7, that

$$a_n(t) \approx b_n(t), \quad \forall n \in \mathbf{N}, \forall \text{ finite } t \text{ in } *J. \quad (19)$$

By internality, we obtain an infinite integer ω such that

$$a_n(t) \approx b_n(t), \quad \forall n \leq \omega, \forall \text{ finite } t \text{ in } *J. \quad (20)$$

Let $p(t) = b_n(t) - b_m(t)$ and $w(t) = (p(t), B^{-1}(t)p(t))$, for n, m infinite. Since every finite t in $*J$ is in $*T$ for some compact interval T with s in $*T$ it follows transferring the Technical Lemma that

$$w(t) \approx 0 \quad \text{for all finite } t \text{ in } *J. \quad (21)$$

Now suppose g is in $D(H^{1/2})$. Then

$$|(p(t), g)| = |(B^{-1/2}(t)p(t), B^{1/2}(t)g)| \leq w(t)^{1/2} c((H+1)g, g)^{1/2}.$$

Therefore from (21) there follows

$$(p(t), g) \approx 0 \quad (22)$$

for all finite t in $*J$ and g in $D(H^{1/2})$. Since $D(H^{1/2})$ is dense in \mathcal{H} and since $\|p(t)\|_{+1}$ is finite for all finite t in $*J$ we conclude that $(p(t), h) \approx 0$ for all h in \mathcal{H} .

That is

$$b_n(t) \underset{\omega}{\approx} b_m(t) \quad (23)$$

for all infinite n and m and finite t in *J . By transfer of Theorem 8 and its proof we know that for each t in *J

$$\text{weak limit}_{\substack{n \rightarrow \infty \\ n \text{ in } {}^*\mathbb{N}}} b_n(t) = b(t).$$

From (23) we may conclude

$$b_m(t) \underset{\omega}{\approx} b(t) \quad (24)$$

for all infinite m and finite t in *J . An analogous argument shows

$$a_m(t) \underset{\omega}{\approx} a(t) \quad (25)$$

for all finite m and finite t in *J .

Combining (16), (24) and (25) we may write $a(t) \underset{\omega}{\approx} a_n(t) \approx b_n(t) \underset{\omega}{\approx} b(t)$ for all finite t in *J and infinite, though sufficiently small, n . In other words

$$U_A(t, s)x_s \underset{\omega}{\approx} U_B(t, s)x_s \quad (26)$$

for all finite t, s in *J and x_s in $D(H^{1/2})$.

Since U_A and U_B are unitary we conclude

$$U_A(t, s)y \underset{\omega}{\approx} U_B(t, s)y \quad (27)$$

for all finite t, s in *J and y in \mathfrak{H} . Again, since the U_A 's and U_B 's are unitary we may argue that

$$\|U_A y - U_B y\|^2 = (U_A y - U_B y, U_A y - U_B y) = 2\|y\|^2 - 2\|y\|^2 = 0.$$

Thus $U_A(t, s)y \approx U_B(t, s)y$ for all finite t, s in *J and y in \mathfrak{H} . Since $\|U_A\| = 1 = \|U_B\|$ we conclude $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in *J . Q.E.D.

PROOF OF COROLLARY 10. Fix $\varepsilon > 0$, $t > s$ and x in $D(H^{1/2})$. Let T be a compact interval containing t and s . Choose K_T as in (11), (12) and (13). Choose N to be an integer larger than $8K_T^2(e^{c(t-s)} - 1)/2\varepsilon c$. Let $m > N$. For any n , let $B_m(n)$ be the m th Yoshida approximation to A_n and C the m th Yoshida approximation to A . Then by the Technical Lemma and the triangle inequality

$$\|A^{-1/2}(t)(U_{A_n}(s, t) - U_A(s, t))x\| \leq 2\varepsilon + \|A^{-1/2}(t)(U_{B_m(n)} - U_C)x\|.$$

Since for any fixed m , $\|B_m(n)\| \leq m$, $\|C\| \leq m$ and $B_m(n)$ converges strongly to C as $n \rightarrow \infty$ it follows from Theorem 7 and the remark which follows it that for n sufficiently large $\|A^{-1/2}(t)(U_{A_n}(s, t) - U_A(s, t))x\| < 3\varepsilon$. Since $D(A^{1/2}(\varepsilon)) = D(H^{1/2})$ is dense it follows that $U_{A_n}(s, t)x \rightarrow U_A(s, t)x$ weakly as $n \rightarrow \infty$ for all $x \in D(H^{1/2})$. Since, in general, weak convergence on a dense set implies strong convergence of unitary operators the proof is completed. Q.E.D.

REMARKS. 1. In the last section of this paper we apply the above results to the analysis of form bounded perturbations. Proofs similar to those just given lead to results useful in studying operator bounded perturbations. Here we just state these results.

DEFINITION. A *Y-generator* on J is a function $A(\cdot)$ from J into the selfadjoint operators on \mathcal{H} such that there are positive numbers E and M satisfying

- (i) $A(t) \geq E + 1$;
- (ii) $t \rightarrow (A(t) + E)^{-1}$ is strongly differentiable with derivative $G(t)$;
- (iii) G is strongly continuous; and
- (iv) $\|(A(t) + E)B(t)\| \leq M$.

Then according to a theorem of Yoshida [15, p. 429] there is a unitary propagator U_A such that for all x in $D(H(s))$, defining $a(t) = U(t, s)x$ gives a solution to the Schrodinger equation in the sense that $(da/dt)(t) = -iA(t)a(t)$. The techniques given above lead to the following theorem and corollary.

THEOREM. Let A be a *Y-generator* with constants E and M . Let B be an internal *Y-generator* with constants E and M . If $(A(t) + E)^{-1} \approx (B(t) + E)^{-1}$ for all finite t in $*J$ then $U_A(t, s) \approx U_B(t, s)$ for all finite t, s in $*J$.

COROLLARY. Let A, A_n be *Y-generators* with constants E, M for $n = 1, 2, \dots$. Let $A_n(t)$ converge to $A(t)$ in the strong resolvent topology for each t in J . Then $U_{A_n}(t, s)$ converges strongly $U_A(t, s)$ for each t, s in J .

2. In Corollary 10 and the last corollary, the convergence of propagators is uniform in any finite interval of J .

IV. Perturbations of a time independent generator. In this section H_0 will be a nonnegative selfadjoint operator on \mathcal{H} . If F is a Hermitian form $Q(F)$ will denote the form domain of F . In particular $Q(H_0) = \mathcal{D}(H_0^{1/2})$.

DEFINITION. A *K-perturbation* of H_0 is a Hermitian form valued function V defined on J with the properties

$$(i) \quad Q(H_0) \subset Q(V(t)). \quad (33)$$

(ii) There are constants $0 \leq a < 1, 0 \leq b < \infty$ such that

$$|V(t)(f, f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b\|f\|^2 \quad (34)$$

for all f in $Q(H_0)$.

(iii) There is a Hermitian form, $\dot{V}(t)$, which is the derivative of V in the sense that $(H_0 + 1)^{-1/2}V(t)(H_0 + 1)^{-1/2}$ is norm differentiable and its derivative is $(H_0 + 1)^{-1/2}\dot{V}(t)(H_0 + 1)^{-1/2}$ and

$$|\dot{V}(t)(f, f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b\|f\|^2 \quad (35)$$

for all f in $Q(H_0)$.

(a, b) are the H_0 -bounds of V .

The following theorem is due to Simon. He states it for the case where each $V(t)$ is a selfadjoint operator but the proof he gives works just as well for each $V(t)$ being a Hermitian form.

THEOREM 11 [11, p. 66]. Let V be a *K-perturbation* of H_0 . There is a constant E such that $A(t) = H_0 + V(t) + E$ is a *K-generator* with constants (H_0, c) for some real number c .

THEOREM 12. Let V, W_n be K -perturbations of H_0 , for $n = 1, 2, \dots$. Suppose the H_0 -bounds of W_n , (a_n, b_n) satisfy $a_n + b_n \rightarrow 0$ as $n \rightarrow \infty$. Then there is a constant E such that for n sufficiently large $A_n(t) = H_0 + V(t) + W_n(t) + E$ is a K -generator and $U_{A_n}(t, s)$ converges strongly to $U_{H_0+V+E}(t, s)$ for all t, s in J .

PROOF. First note that $Q(H_0) \subset Q(V(t)) \cap Q(W_n(t)) = Q(V(t) + W_n(t))$. Next observe that

$$\begin{aligned} |(V(t) + W_n(t))(f, f)| &\leq |V(t)(f, f)| + |W_n(t)(f, f)| \\ &\leq a(H_0^{1/2}f, H_0^{1/2}f) + b(f, f) + a_n(H_0^{1/2}f, H_0^{1/2}f) + b_n\|f\|^2 \\ &\leq ((a+1)/2)(H_0^{1/2}f, H_0^{1/2}f) + 2b\|f\|^2 \end{aligned}$$

for all f in $Q(H_0)$ and n sufficiently large. Also, letting $K = (H_0 + 1)^{-1/2}$, it is true that $K(V(t) + W_n(t))K = KV(t)K + KW_n(t)K$ and so $(d/dt)K(V(t) + W_n(t))K$ exists in norm and equals $K\dot{V}(t)K + K\dot{W}_n(t)K$. As above,

$$|(\dot{V}(t) + \dot{W}_n(t))(f, f)| \leq ((a+1)/2)(H_0^{1/2}f, H_0^{1/2}f) + 2b\|f\|^2.$$

By Theorem 11, there is a constant E such that for n sufficiently large $A_n(t)$ is a K -generator.

$$|(KW_n(t)Kf, f)| \leq (a_n + b_n)\|f\|^2$$

so $\|KW_n(t)K\| \rightarrow 0$ as $n \rightarrow \infty$, uniformly in t . Consequently, by the Neumann series for the resolvent, one obtains that $(1 + K(V(t) + E)K + KW_n(t)K)^{-1}$ converges in norm to $(1 + K(V(t) + E)K)^{-1}$ as $n \rightarrow \infty$. We conclude that $(A_n(t) + 1)^{-1} = K(1 + K(V(t) + E)K + KW_n(t)K)^{-1}K$ converges in norm, and hence strongly, to $(H_0 + V(t) + E + 1)^{-1}$. We are done by Corollary 10. Q.E.D.

As a special case of the previous theorem we note

COROLLARY 13. Let V, W be K -perturbations of H_0 . There is a constant E so that $A_\varepsilon(t) = H_0 + V(t) + \varepsilon W(t) + E$ is a K -generator for all ε sufficiently small and $U_{A_\varepsilon}(t, s)$ converges strongly to $U_A(t, s)$ for all t, s in J where $A = H_0 + V(t) + E$ as $\varepsilon \rightarrow 0$.

EXAMPLE 14. Let V be a small Hermitian form perturbation of H_0 , i.e., $Q(H_0) \subset Q(V)$ and $|V(f, f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b\|f\|^2$ for all $f \in Q(H_0)$ where $0 \leq a < 1$ and $0 \leq b < \infty$. Let f be a bounded C^1 function with bounded derivative. Let $A_\varepsilon(t) = H_0 + \varepsilon f(t)V + E$. Then $U_{A_\varepsilon}(t, s)$ converges strongly to $U_{H_0+E}(t, s)$ as $\varepsilon \rightarrow 0$.

EXAMPLE 15. Let H_0 here denote $-\Delta$ on $L^2(\mathbb{R}^n)$. Let τ be a real distribution on \mathbb{R}^n . τ is called an H_0 -small distribution if there are numbers $0 \leq a < 1$, $0 \leq b < \infty$ such that

$$\pm \tau(\bar{f}f) \leq a(H_0^{1/2}f, H_0^{1/2}f) + b\|f\|^2 \quad (36)$$

for all f in $C_c^\infty(\mathbb{R}^n)$. (a, b) are the H_0 -bounds of τ . Given such a distribution one can define a Hermitian form $\tau(f, g) = \tau(\bar{f}g)$ which can be extended to a Hermitian form τ on $Q(H_0)$ satisfying

$$|\tau(f, f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b(f, f) \quad (37)$$

for all f in $Q(H_0)$.

Let ρ be a $C_c^\infty(\mathbf{R}^n)$ function which satisfies $\rho(x) = 1$ for $\|x\| \leq 1$ and $0 \leq \rho \leq 1$. Set $\beta_k(x) = \rho(x/k)$ and $\gamma_n(x) = \rho(kx)(\int \rho(kx) dx)^{-1}$. Define $V_k(f, g) = \tau(\gamma_n * (\beta_k \bar{f}g))$, so that each V_k is a $C_c^\infty(\mathbf{R}^n)$ function. The V_k 's are the regularizations of τ and converge to τ in the sense of distributions. Moreover $H_0 + V_k$ converges in the strong resolvent topology to the form sum $H_0 + \tau$. See [3] for details.

Suppose f is C^1 on J . Define $\|f\| = \max\{\|f\|_\infty, \|df/dt\|_\infty\}$. Suppose $\|f\| < 1/a$.

Let $A(t) = H_0 + f(t)\tau$ and $A_k(t) = H_0 + f(t)V_k$. Then $f(t)V$ and $f(t)V_k$ are K -perturbations of H_0 and by the preceding remarks $A_k(t)$ converges in the strong resolvent topology to $A(t)$ for each t on J . By Theorems 11, 9 and Corollary 10 $U_{A_k}(t, s)$ converges strongly to $U_A(t, s)$ for each t, s in J while $U_{A_k}(t, s) \approx U_A(t, s)$ for each infinite k in ${}^*\mathbf{N}$ and every finite t, s in *J .

For specific examples of τ 's which may be chosen we refer the reader to [3]. Here we note that τ may be in $L^p(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ where $p \geq 1$ if $n = 1$, $p > 1$ if $n = 2$ and $p \geq n/2$ if $n \geq 3$. Also τ may be a delta function concentrated on a C^1 compact hypersurface or the distributional directional derivative of a bounded function.

If τ_1 and τ_2 are two H_0 -small distributions then for suitable f 's $V(t) = f(t)\tau_1 + (1 - f(t))\tau_2$ is a K -perturbation of H_0 . Choosing f so that $f = 0$ for $t \leq 0$ and $f = 1$ for $t > 1$ one can treat the problem of switching from one potential to another. By choosing $f(t) = \sin wt$ one obtains periodically varying potentials.

One obtains considerable technical advantage in continuing the study of time dependent singular perturbations of $-\Delta$ by introducing infinite regularizations and by accepting infinitesimal errors. This naturally occurs because $U_{H_0+V_k+E}$ has the generator $H_0 + V_k + E$ defined as an operator sum with V_k in ${}^*C_c^\infty$ while U_{H_0+V+E} has the generator $H_0 + V + E$ defined as a form sum.

For example, since $V_k \in {}^*L^p(\mathbf{R}^n)$ for all $p \geq 1$, the product formula of Faris [2] is valid and one can express the solution u of the Schrodinger equation (1) as

$$u(t) \approx \prod_{j=0}^{m-1} \left(\exp\left(\frac{it}{m}\Delta\right) \exp - \frac{it}{m} V_k\left(\frac{jt}{m}\right) \right) u_s \quad (38)$$

with $u(0) = u_s$ for u_s in $Q(H_0)$, for m sufficiently large in ${}^*\mathbf{N}$. Since $L^2(\mathbf{R}^n)$ is separable one m can be chosen so that (38) holds for all u_s in $Q(H_0)$ and all finite t (see [13] for details of this argument). Following Faris, (38) leads to a Feynman path integral:

For each pair of points (x_0, x_m) in ${}^*\mathbf{R}^{2n}$ we may view ${}^*\mathbf{R}^{n(m-2)}$ as a space of "polygonal paths" connecting x_0 and x_m . These pairs are not standard, but if x_0 and x_m are finite and if ω is any standard path connecting x_0 and x_m , i.e. $\omega: [0, t] \rightarrow \mathbf{R}^n$ and $\omega(0) = \text{st}(x_0)$, $\omega(t) = \text{st}(x_m)$, then there are polygonal paths ω^1 so that $\omega(s) \approx \omega^1(s)$ for all s in $[0, t]$. There are many more polynomial paths than standard paths. Since for each t , $V_k(t)$ is a ${}^*C_c^\infty$ function we let $(V_k(t))(y) = V_k(y, t)$. For each polygonal path $x = (x_1, \dots, x_{m-1})$ in ${}^*\mathbf{R}^{n(m-2)}$ let

$$S(x_0, x, x_m; m, t, V) = \sum_{j=1}^m \left[\frac{(x_j - x_{j-1})^2}{2(t/m)^2} - V_k\left(x_j, \frac{jt}{m}\right) \right] \frac{t}{m}.$$

Let $u(\cdot, t) \in L^2(\mathbf{R}^n)$ be the state at time t of the quantum mechanical system with time dependent Hamiltonian $H_0 + V(t)$ with state at time 0 given by $u(\cdot, 0) \in Q(H_0)$. Thus $(U_{H_0+V}(t, 0))(u(\cdot, 0)) = u(\cdot, t)$. Define the amplitude of the corresponding quantum mechanical particle to go from x_m at time 0 to x_0 at time t by

$$K((x_0, t), (x_m, 0)) = \int_{\mathbf{R}^{(m-2)n}} \exp(iS(x_0, x, x_m; m, t, V_k)) u(x_m, 0) dx. \quad (39)$$

Since $e^{is\Delta}$ is convolution by $(4\pi is)^{-3n/2} e^{iy^2/4s}$ in \mathbf{R}^n , (38) implies

$$u(\cdot, t) \approx \int_{\mathbf{R}^n} K((\cdot, t), (x_m, 0)) u(x_m, 0) dx_m \quad (40)$$

in $*L^2(\mathbf{R}^n)$.

(39) and (40) are a formulation of the Feynman path integral for particles moving in singular and time dependent potentials.

Since adding a delta function to $-\Delta = -d^2/dy^2$ in $L^2(\mathbf{R}^1)$ is equivalent to imposing a boundary condition on the maximally defined $-d^2/dy^2$, it follows that the problem of smoothly changing boundary conditions in time can be given a Feynman path integral interpretation.

We conclude the discussion of this example by noting that, in general, the solution to equation (1) is shown to exist by proving certain approximate solutions converge weakly. However, by accepting infinitesimal errors one can explicitly construct the propagator. For this purpose we enter the interaction representation defining $\tilde{V}(t) = e^{itH_0} V_k(t) e^{-itH_0}$, where k remains an infinite positive integer. Then $t \rightarrow \tilde{V}(t)$ is an integral D -generator so that

$$\tilde{U}(t, s) = 1 + \sum_{p=1}^{\infty} (-i)^p \int_s^t \int_s^{t_1} \cdots \int_s^{t_{p-1}} \tilde{V}(t_1) \cdots \tilde{V}(t_p) dt_p \cdots dt_1$$

converges in norm to a unitary propagator and $U_a(t, s) = e^{-itH_0} \tilde{U}(t, s) e^{isH_0}$ is almost the propagator for equation (1) in the sense that $U(t, s)x_s \approx U_a(t, s)x_s$ for all x_s in $Q(H_0)$ and t, s finite. For the standard details, see [10]. Thus we have defined time ordered exponentials of certain quadratic form valued functions.

REFERENCES

1. P. Chernoff, *Product formulas, nonlinear semigroups and addition of unbounded operators*, Mem. Amer. Math. Soc. No. 140 (1974), 1-121.
2. W. Faris, *Product formulas for perturbations of linear propagators*, J. Functional Analysis **1** (1967), 93-108.
3. I. Herbst and A. Sloan, *Perturbation of translation invariant positivity preserving semigroups on $L^2(\mathbf{R}^n)$* , Trans. Amer. Math. Soc. **236** (1978), 325-360.
4. T. Kato, *Integration of the equation of evolution in a Banach space*, J. Math. Soc. Japan **5** (1953), 208-234.
5. ———, *Perturbation theory for linear operators*, Springer, Berlin, 1966.
6. ———, *On linear differential equations in Banach spaces*, Comm. Pure Appl. Math. **9** (1956), 479-486.
7. ———, *Linear evolution equations of hyperbolic type*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **17** (1970), 241-258.
8. J. Kisynski, *Sur les opérateurs de Green des problèmes de Cauchy abstraits*, Studia Math. **23** (1964), 285-328.
9. E. Nelson, *Internal set theory: A new approach to nonstandard analysis*, Bull. Amer. Math. Soc. **83** (1977), 1165-1198.

10. M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, selfadjointness*, Academic Press, New York, 1975.
11. B. Simon, *Quantum mechanics for Hamiltonians defined as quadratic forms*, Princeton Univ. Press, Princeton, N. J., 1971.
12. A. Sloan, *An application of the nonstandard Trotter product formula*, J. Mathematical Phys. **18** (1977), 2495–2496.
13. ———, *A note on exponentials of distributions*, Pacific J. Math. **79** (1978), 207–224.
14. K. Stroyan and W. Luxemburg, *Introduction to the theory of infinitesimals*, Academic Press, New York, 1976.
15. K. Yoshida, *Functional analysis*, Springer-Verlag, Berlin, 1965.
16. Giuseppe DaPrato and Mimmo Iannelli, *On a method for studying abstract evolution equations in the hyperbolic case*, Comm. Partial Differential Equations **1** (1976), 585–608.

DEPARTMENT OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30329